On Weitzenböck Curvature Operators

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Abstract

The Weitzenböck curvature operators are the curvature terms of order zero that appear in the well known classical Weitzenböck formula. In this paper, we use the formalism of double forms to prove a simple formula for this operators and to study their geometric properties.

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1 Introduction

The Weitzenböck curvature operator, denoted throughout this paper by $\mathcal{N}$, is the curvature term of order zero (i.e. depends linearly on the Riemann curvature tensor) that appears in the well known classical Weitzenböck formula. The former expresses the Laplacian Δ of differential forms in terms of the Levi-Civita connexion ∇, precisely we have:

$$\Delta = \nabla^*\nabla + \mathcal{N}.$$

This formula is particularly important in the study of interactions between the geometry and topology of manifolds. In fact, there exists a method, due to Bochner and known as vanishing theorems, consisting of proving the vanishing of Betti numbers of a Riemannian manifold with a positive curvature condition stronger than the positivity of $\mathcal{N}$. This method mainly applies to compact manifolds.

For each p, the curvature operator $\mathcal{N}$ preserves p-forms and it is self-adjoint. Therefore, by duality, we can consider it as a double form.

The main obstacle with $\mathcal{N}$ is that it is actually a complicated expression of the curvature. Several simplifications of such expression exist in the literature. For instance, the Clifford formalism of Lawson-Michelsohn [10], see

also [4], the works of Gallot-Meyer [2], Maillot [11], Bourguignon [1] ...
In this paper, using the formalism of double forms, we prove a simple formula for $\mathcal{N}$ and then we use it to study some geometric properties of this curvature. This formula was first established by Bourguignon, see proposition 8.6 in [1]. Unfortunately, up to my knowledge, this nice formula is not well known and it is not used even though the paper [1] is very famous!. From my side, I noticed the existence of this formula only once my proof was finished. However our proof is completely different. It is of algebraic nature and direct.

2 Double Forms

Let (V, g) be an Euclidean real vector space of dimension n. In the following we shall identify whenever convenient (via their Euclidean structures), the vector spaces with their duals. Let $\Lambda V = \bigoplus_{p\geq 0} \Lambda^p V$ denotes the exterior algebra of p-vectors on V.
A double form on V of degree (p, q) can be defined as a bilinear form $\Lambda^p V \times \Lambda^q V \to \mathbf{R}$. That is a multilinear form which is skew symmetric in the first p-arguments and also in the last q-arguments. If $p = q$ and the bilinear form is symmetric we say that we have a symmetric double form.
Double forms are abundant in geometry. The Riemann curvature tensor and the Weyl curvature tensor are symmetric double forms of order $(2, 2)$. The metric, Ricci, Einstein, and Schouten tensors are symmetric double forms of degree $(1, 1)$. Gauss-Kronecker tensors [13] and the Weitzenböck curvatures (see the next section) are examples of symmetric double forms of higher order.
The exterior product between p-vectors extends in a natural way to double forms of any degree and we obtain the so called Kulkarni-Nomizu product. For a (p, q)-double form ω_1 and an (r, s)-double form ω_2, the product $\omega_1\omega_2$ is a double form of degree $(p + r, q + s)$ given by

$$\begin{aligned}
&\omega_1.\omega_2(x_1 \wedge ... \wedge x_{p+r}, y_1 \wedge ... \wedge y_{q+s})\\
&= \frac{1}{p!r!s!q!} \sum_{\sigma\in S_{p+r},\rho\in S_{q+s}} \epsilon(\sigma)\epsilon(\rho)\omega_1(x_{\sigma(1)} \wedge ... \wedge x_{\sigma(p)}, y_{\rho(1)} \wedge ... \wedge y_{\rho(q)})\\
&\qquad\qquad \omega_2(x_{\sigma(p+1)} \wedge ... \wedge x_{\sigma(p+r)}, y_{\rho(q+1)} \wedge ... \wedge y_{\rho(q+s)})
\end{aligned} \tag{1}$$

In particular , the product of the inner product g with itself k-times determines the canonical scalar product on $\Lambda^p V$. Precisely we have

$$g^k(x_1 \wedge ... \wedge x_k, y_1 \wedge ... \wedge y_k) = k! \det[g(x_i, y_j)].$$

Let us denote by $S^2\Lambda^p V$ the set of all the symmetric double forms of order (p,p) on V and let $S^2\Lambda V = \bigoplus_{p=0}^{p=n} S^2\Lambda^p V$. Recall that it is a commutative algebra.

2.1 The multiplication map by g^k and the contraction map

The multiplication map by the powers of the metric, that is g^k, in $S^2\Lambda^p V$ plays an important role in the study of double forms. In [9] we proved the following fundamental property of this map:

Proposition 2.1 ([9]) *Let ω_1, ω_2 be two (p,p)-double forms and $k \leq n-2p$, then*

$$g^k\omega_1 = g^k\omega_2 \Longrightarrow \omega_1 = \omega_2. \tag{2}$$

Note here for future reference that, for each p, the natural scalar product $\frac{g^p}{p!}$ on $\Lambda^p V$ induces canonically a natural scalar product on $S^2\Lambda^p V$, we shall denote it by $\langle , \rangle$. We extend $\langle , \rangle$ to $\bigoplus_{p=0}^{p=n} S^2\Lambda^p V$ by declaring that $S^2\Lambda^p V \perp S^2\Lambda^q V$ if $p \neq q$.
A second fundamental map on double forms is the contraction map c. It decreases the order of a double form by 1 and it is the adjoint of the multiplication map by g, precisely we have.

Proposition 2.2 ([9]) *For arbitrary double forms ω_1, ω_2 we have*

$$\langle g\omega_1, \omega_2 \rangle = \langle \omega_1, c\omega_2 \rangle. \tag{3}$$

In particular, for every $k \geq 1$, we have $\langle g^k\omega_1, \omega_2 \rangle = \langle \omega_1, c^k\omega_2 \rangle$. Where c^k denotes the composition of the contraction map c with itself k-times.

2.2 The first Bianchi identity and sectional curvatures

We say that a double form ω of degree (p,q), with $q \geq 1$, satisfies the first Bianchi identity if for all vectors $(x_i), (y_j)$ we have

$$\sum_{j=1}^{p+1} (-1)^j \omega(x_1 \wedge ... \wedge \hat{x}_j \wedge ... x_{p+1}, x_j \wedge y_1 \wedge ... \wedge y_{q-1}) = 0,$$

where ˆ denotes omission. Note that this identity is preserved by multiplication of double forms. Let us then denote by $S_1^2(\Lambda V)$ the sub-algebra of $S^2(\Lambda V)$ consisting of all symmetric double forms satisfying the first Bianchi identity. This space is not irreducible under the natural action of the orthogonal group. In fact, Kulkarni [3] proved that the full reduction into irreducible components is given by

$$S_1^2(\Lambda^p V) = E_1^p \oplus gE_1^{p-1} \oplus g^2 E_1^{p-2} \oplus ... \oplus g^p E_1^0. \tag{4}$$

Where, for each $0 \leq k \leq p$, $E_1^k = \{\omega \in S_1^2(\Lambda^k V) : c\omega = 0\}$.
Another important property of double forms satisfying the first Bianchi identity is that they are determined by their sectional curvatures. Recall that the sectional curvature of a given symmetric (p,p)-double form ω is a function, say K_ω, defined on the Grassman algebra of p-planes in V. For a p-plane P, we set

$$K_\omega(P) = \omega(e_1 \wedge ... \wedge e_p, e_1 \wedge ... \wedge e_p),$$

where $\{e_1, ..., e_p\}$ is any orthonormal basis of P. We have the following characterization of symmetric double forms satisfying the first Bianchi identity and with constant sectional curvature [3]:

$$K_\omega \equiv c \quad \text{is constant} \quad \text{if and only if} \quad \omega = c\frac{g^p}{p!}. \tag{5}$$

For a symmetric (r,r)-double form ω and every p, $0 \leq p \leq n-r$, we can use formula (1) to evaluate the sectional curvature of the products $g^p\omega$, this will be used later in this paper. Let $\{e_1, ..., e_{p+r}\}$ be orthonormal vectors, then

$$\begin{aligned} &g^p\omega(e_1 \wedge ... \wedge e_{p+r}, e_1 \wedge ... \wedge e_{p+r}) \\ &= p! \sum_{1 \leq i_1 < i_2 < ... < i_r \leq p+r} \omega(e_{i_1} \wedge ... \wedge e_{i_r}, e_{i_1} \wedge ... \wedge e_{i_r}). \end{aligned} \tag{6}$$

2.3 Generalized Hodge star operator

We suppose here that an orientation is fixed on the vector space V. The classical Hodge star operator $* : \Lambda^p V \to \Lambda^{n-p} V$ can be extended in a natural way to double forms. For a double form ω, set $*\omega(.,.) = \omega(*.,*.)$. Many classical properties can be generalized to the case of this new operator, see [9]. The generalized Hodge star operator was shown to be an important tool in the study of double forms. In particular, it provides a second simple

relation between the contraction map c and the multiplication map by g, as follows:

Proposition 2.3 ([9]) *For every (p,p)-double form ω, we have*

$$g\omega = *c * \omega. \tag{7}$$

In particular, for every $k \geq 1$, we have $g^k\omega = *c^k * \omega$.

3 Weitzenböck Operators

3.1 Clifford multiplication of p-vectors

Let $e \in V$, recall that the interior product i_e on ΛV is the adjoint of the exterior multiplication map by e. We define the Clifford multiplication of two p-vectors, denoted by a dot, as follows: For $e \in V$ and $\omega \in \Lambda V$, set

$$e.\omega = e \wedge \omega - i_e\omega.$$

In particular, for $e, f \in V$ we have $e.f = e \wedge f - g(e,f)$.
Assuming this product to be associative, we can define the products $e_{i_1}.....e_{i_p}$ for all p and $e_{i_k} \in V$. Then using linearity, we can extend it to a product on ΛV: the Clifford product.
It is not difficult to check that we can recover the exterior product as:

$$e_{i_1} \wedge ... \wedge e_{i_p} = \frac{1}{p!} \sum_{\sigma \in S_p} \epsilon(\sigma) e_{\sigma(i_1)}..... \wedge e_{\sigma(i_p)}. \tag{8}$$

3.2 Definition of Weitzenböck operators

For $\phi \in \Lambda^2 V$, we define the linear operator ad_ϕ on ΛV by

$$ad_\phi(\psi) = [\phi, \psi] = \phi.\psi - \psi.\phi.$$

In particular, a straightforward computation shows that for orthonormal vectors $\{e_i\}$ we have

$$ad_{e_i.e_j} e_{i_1}.....e_{i_p} = \begin{cases} 0 & \text{if} \quad i,j \in \{i_1,...,i_p\} \quad \text{or} \quad i,j \notin \{i_1,...,i_p\} \\ 2e_i.e_j.e_{i_1}.....e_{i_p}, & \text{otherwise.} \end{cases} \tag{9}$$

Let now ω be a symmetric $(2,2)$-double form. We define the Weitzenböck transformation of order p at ω to be the symmetric (p,p)-double form $\mathcal{N}_p(\omega)$ defined as follows, see [10, 5, 4].

$$\mathcal{N}_p(\omega)(\psi_1,\psi_2)=\frac{1}{4}\sum_{i<j,k<l}\omega(e_i\wedge e_j,e_k\wedge e_l)\langle ad_{e_i.e_j}\psi_1,ad_{e_k.e_l}\psi_2\rangle. \qquad (10)$$

Where $(e_1,...,e_n)$ denotes an arbitrary orthonormal basis of V.
This definition is of course motivated by the curvature term in the Weitzenböck formula. Note that $\mathcal{N}_p$ is a linear operator.

3.3 Sectional curvatures of Weitzenböck

Let P be a p-plane in V spanned by orthonormal vectors $\{e_1,...,e_p\}$. A direct computation using formula (9) shows that

$$\begin{aligned}
&\mathcal{N}_p(\omega)(e_1\wedge...\wedge e_p,e_1\wedge...\wedge e_p)\\
&\quad=\frac{1}{4}\sum_{k<l}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\omega(e_i\wedge e_j,e_k\wedge e_l)(-1)^{i+1}\langle 2e_j.e_1...\hat{e_i}...e_p,ad_{e_k.e_l}e_1...e_p\rangle\\
&\quad=\frac{1}{4}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\omega(e_i\wedge e_j,e_i\wedge e_j)\langle 2e_j.e_1...\hat{e_i}...e_p,2e_j.e_1...\hat{e_i}...e_p\rangle\\
&\quad=\sum_{i=1}^{p}\sum_{j=p+1}^{n}\omega(e_i\wedge e_j,e_i\wedge e_j).
\end{aligned} \qquad (11)$$

Now using (6) we get for $p\geq 2$:

$$\begin{aligned}
\sum_{i=1}^{p}\sum_{j=p+1}^{n}\omega(e_i\wedge e_j,e_i\wedge e_j)&=\sum_{i=1}^{p}\left\{c\omega(e_i,e_i)-\sum_{i,j=1}^{p}\omega(e_i\wedge e_j,e_i\wedge e_j)\right\}\\
&=\{\frac{g^{p-1}}{(p-1)!}c\omega-2\frac{g^{p-2}}{(p-2)!}\omega\}(e_1\wedge...\wedge e_p,e_1\wedge...\wedge e_p).
\end{aligned} \qquad (12)$$

We conclude that the two double forms $\mathcal{N}_p$ and $\{\frac{g^{p-1}}{(p-1)!}c\omega-2\frac{g^{p-2}}{(p-2)!}\omega\}$ have the same sectional curvatures for every $2\leq p\leq n-2$. It is then natural to expect the equality of these two double forms. This is in fact true and shall be proved in the section below.

3.4 A simple formula for the Weitzenböck Transformation

In the proposition below, we prove a simple formula for the transformation $\mathcal{N}$.

Theorem 3.1 *Let ω be a symmetric $(2,2)$-double form satisfying the first Bianchi identity, then for every $2 \leq p \leq n-2$, the Weitzenböck transformation of order p is determined by*

$$\mathcal{N}_p(\omega) = \left\{ \frac{gc\omega}{p-1} - 2\omega \right\} \frac{g^{p-2}}{(p-2)!}. \tag{13}$$

Proof. We shall prove the equality by evaluating both terms of the equation on decomposed p-vectors. Let $\{e_1, ..., e_n\}$ be an arbitrary orthonormal basis of V and $p \geq 1$. Let $e_I = e_{i_1} \wedge ... \wedge e_{i_p}$ and $e_J = e_{j_1} \wedge ... \wedge e_{j_p}$ be two arbitrary elements of the standard basis of $\Lambda^p V$.
First, it is straightforward from formulas (1) and (10) that if $\mathrm{card}(I \cap J) \leq p-3$ then

$$\mathcal{N}_p(\omega)(e_I, e_J) = \left\{ \frac{gc\omega}{p-1} - 2\omega \right\} \frac{g^{p-2}}{(p-2)!}(e_I, e_J) = 0.$$

Next, suppose $\mathrm{card}(I \cap J) = p-2$. Without loss of generality, we may re-index as follows: $e_I = e_1 \wedge ... \wedge e_p$ and $e_J = e_1 \wedge ... \wedge e_{p-2} \wedge e_{p+1} \wedge e_{p+2}$. Let $f_I = f_1 \wedge ... \wedge f_p = e_J$, where

$$\begin{cases} f_j = e_j & \text{if} \quad j < p-1 \\ f_{p-1} = e_{p+1} & \text{and} \quad f_p = e_{p+2} \end{cases}$$

Let $\omega_1 = \frac{1}{(p-2)!}\{\frac{gc\omega}{(p-1)} - 2\omega\}$, then using formula (1) for the product, the right hand side of equation (13) is given by:

$$g^{p-2}\omega_1(e_I, e_J) = \sum_{\sigma,\rho \in Sh(2,p-2)} \epsilon(\sigma)\epsilon(\rho)\omega_1(e_{\sigma(1)} \wedge e_{\sigma(2)}, f_{\rho(1)} \wedge f_{\rho(2)})$$
$$g^{p-2}(e_{\sigma(3)} \wedge ... \wedge e_{\sigma(p)}, f_{\rho(3)} \wedge ... \wedge f_{\rho(p)}). \tag{14}$$

Where $\sigma \in Sh(2,p)$ means a permutation of $\{1, ..., p\}$ such that $\sigma(1) < \sigma(2)$ and $\sigma(3) < ... < \sigma(p)$.
Remark that the only permutations which yield non zero terms in the previous summation are

$$\sigma = \rho = \begin{pmatrix} 1 & 2 & 3 & ... & p \\ p-1 & p & 1 & ... & p-2 \end{pmatrix}$$

Since $\epsilon(\sigma)=\epsilon(\rho)=1$ we get

$$\begin{aligned} g^{p-2}\omega_1(e_I,e_J) &= (p-2)!\omega_1(e_{p-1}\wedge e_p, e_{p+1}\wedge e_{p+2}) \\ &= -2\omega(e_{p-1}\wedge e_p, e_{p+1}\wedge e_{p+2}). \end{aligned}$$

On the other hand, using (9) we have

$$\begin{aligned} 4\mathcal{N}_p(\omega)(e_I,e_J) = \sum_{k<l}\sum_{i=1}^{p}\sum_{j=p+1}^{n} &\omega(e_i\wedge e_j, e_k\wedge e_l)(-1)^{i+1} \\ &\langle 2e_j.e_1...\hat{e_i}....e_p, ad_{e_k.e_l}e_1.....e_{p-2}e_{p+1}e_{p+2}\rangle. \end{aligned} \tag{15}$$

Note that the terms with $i\leq p-2$ vanish. For $i=p-1$ (resp. for $i=p$) , the only non zero terms are those corresponding to $k=p, l=p+1, j=p+2$ or $k=p, l=p+2, j=p+1$ (resp. $k=p-1, l=p+1, j=p+2$ or $k=p-1, l=p+2, j=p+1$). Therefore the previous summation reduces to

$$\begin{aligned} 4\mathcal{N}_p(\omega)(e_I,e_J) &= 8\omega(e_{p-1}\wedge e_{p+2}, e_p\wedge e_{p+1}) - 8\omega(e_{p-1}\wedge e_{p+1}, e_p\wedge e_{p+2}) \\ &= 8\omega(e_{p-1}\wedge e_p, e_{p+2}\wedge e_{p+1}). \end{aligned} \tag{16}$$

Where in the last equality we have used the the first Bianchi identity.
We now study the next case where $\mathrm{card}(I\cap J)=p-1$. Here also without loss of generality we may assume $e_I=e_1\wedge...\wedge e_p$ and $e_J=e_1\wedge...\wedge e_{p-1}\wedge e_{p+1}$. Let $f_I=f_1\wedge...\wedge f_p=e_J$, where

$$\begin{cases} f_j=e_j & \text{if} \quad j<p \\ f_p=e_{p+1}. \end{cases}$$

The product $g^{p-2}\omega_1(e_I,e_J)$ has the form (14). In this case, the only permutations which yield non zero terms are the following:

$$\sigma_k=\rho_k=\begin{pmatrix} 1 & 2 & 3 & ... & k & ... & p \\ k & p & 1 & ... & \hat{k} & ... & p-1 \end{pmatrix},$$

where k ranges from 1 to $p-1$. Therefore we immediately obtain

$$\begin{aligned} g^{p-2}\omega_1(e_I,e_J) &= (p-2)!\sum_{k=1}^{p-1}\omega_1(e_k\wedge e_p, e_k\wedge e_{p+1}) \\ &= \sum_{k=1}^{p-1}\left\{\frac{1}{p-1}c\omega(e_p,e_{p+1}) - 2\omega(e_k\wedge e_p, e_k\wedge e_{p+1})\right\} \qquad (17) \\ &= c\omega(e_p,e_{p+1}) - 2\sum_{k=1}^{p-1}\omega(e_k\wedge e_p, e_k\wedge e_{p+1}). \end{aligned}$$

We now evaluate the right hand term on the same p-vectors. Recall that $\mathcal{N}_p(\omega)(e_I,e_J)$ can be written as a summation as in (15). In this summation and for $i\leq p-1$ (resp. for $i=p$), the only non zero terms are those corresponding to $k=i, l=p, j=p+1$ (resp. $k=p+1, l=j$). Therefore we can write

$$\begin{aligned} 4\mathcal{N}_p(\omega)(e_I,e_J) &= -4\sum_{i=1}^{p-1}\omega(e_i\wedge e_{p+1}, e_i\wedge e_p) + 4\sum_{j=p+1}^{n}\omega(e_i\wedge e_{p+1}, e_i\wedge e_p) \\ &= 4c\omega(e_p,e_{p+1}) - 8\sum_{i=1}^{p-1}\omega(e_i\wedge e_{p+1}, e_i\wedge e_p). \end{aligned} \qquad (18)$$

Finally, In the case where $\mathrm{card}(I\cap J)=p$, then $I=J$. This case was already discussed in the previous section (that is the two double forms have the same sectional curvatures). This completes the proof of the theorem. ■

As a consequence of the previous theorem, the Weitzenböck transformation preserves the first Bianchi identity. That is, it sends symmetric $(2,2)$-double forms satisfying the first Bianchi identity to symmetric (p,p)-double forms satisfying the first Bianchi identity. Furthermore we have:

Corollary 3.2 *For every $2\leq p\leq n$ the Weitzenböck linear transformation*

$$\mathcal{N}_p : S_1^2(\Lambda^2 V)\to S_1^2(\Lambda^p V)$$

is injective. Furthermore, its adjoint operator is given by

$$\mathcal{N}_p^*\omega = \left(\frac{gc^{p-1}}{(p-1)!} - 2\frac{c^{p-2}}{(p-2)!}\right)\omega$$

Proof. The first claim results directly from the previous theorem and proposition 2.1. The second claim is a direct consequence of proposition 2.2. ■

4 Geometric Properties

In this section, Let ω be a fixed symmetric $(2,2)$-double form satisfying the first Bianchi identity on an n-dimensional vector space V (keeping in mind the typical example of the Riemann curvature tensor of a Riemannian n-manifold at a given point). In order to simplify the notations we shall denote by $\mathcal{N}_p$ the double form $\mathcal{N}_p(\omega)$.
The following result can be proved easily using the previous theorem after noticing that both double forms satisfy the first Bianchi identity and they have the same sectional curvatures (see formula 11) :

Proposition 4.1 *For each p such that $2 \le p \le n-2$, we have*

$$*\mathcal{N}_p = \mathcal{N}_{n-p}. \tag{19}$$

In particular, if $n = 2p$ then $*\mathcal{N}_p = \mathcal{N}_p$.

It is clear from the previous theorem that the double form $\mathcal{N}_p$ is divisible by g^{p-2} and hence its orthogonal decomposition is reduced as follows:

Proposition 4.2 *With respect to the irreducible orthogonal decomposition (4), if $\omega = \omega_2 + g\omega_1 + g^2\omega_0$, then the double form $\mathcal{N}_p$, $2 \le p \le n-2$, splits as follows:*

$$\mathcal{N}_p = g^{p-2}\left\{\frac{-2\omega_2}{(p-2)!}\right\} + g^{p-1}\left\{\frac{(n-2p)\omega_1}{(p-1)!}\right\} + g^p\left\{\frac{2(n-p)\omega_0}{(p-1)!}\right\}.$$

Let now $\omega = R$ be the Riemann curvature tensor of a given Riemannian manifold. Recall that the sectional curvature of Weitzenböck is given by (11) with $\omega = R$, that is a generalization of the Ricci curvature. Also, a double form of order (p,p) and satisfying the first Bianchi identity is with constant sectional curvature if and only if it is proportional to the metric g^p. The following corollary is therefore a straightforward consequence of the previous proposition.

Corollary 4.3 *For $2 \le p \le n-2$, a Riemannian manifold (M,g) of dimension n has its Weitzenböck sectional curvature of order p constant if and only if it is either with constant sectional curvature or conformally flat with dimension $n = 2p$.*

Let us note here that the previous corollary was first proved by Tachibana in [12]. Where the sectional curvature of Weitzenböck is called the mean curvature of a p-plane.

We now go back to the general situation where ω is any algebraic (p,p)-double form. Using formula (12) in [9] and the previous proposition, a long but straightforward computation shows that:

Theorem 4.4 *For every $2 \leq p \leq n-2$, the contractions of the double form $\mathcal{N}_p$ up to the order p and $p-1$ are respectively given by*

$$\begin{aligned} c^p(\mathcal{N}_p) &= \frac{p(n-2)!}{(n-p-1)!}c^2\omega, \\ c^{p-1}(\mathcal{N}_p) &= \frac{(n-3)!}{(n-p-1)!}\left\{(n-2p)c\omega + (p-1)c^2\omega g\right\}. \end{aligned} \tag{20}$$

Furthermore, for any $k \leq p-2$, the contraction up to the order k is given by

$$\begin{aligned} c^k(\mathcal{N}_p) = \frac{(n-p+k-2)!g^{p-k-2}}{(n-p-2)!(p-k-2)!}\{-2\omega &+ \frac{n-k-p-1}{(n-p-1)(p-k-1)}gc\omega \\ &+ \frac{k}{(n-p-1)(p-k-1)(p-k)}g^2c^2\omega\}. \end{aligned} \tag{21}$$

Remark. Let $E = \frac{1}{2}c^2\omega g - c\omega$ be the Einstein curvature tensor of ω, then the second formula in (20) may be written in the following alternative form:

$$c^{p-1}(\mathcal{N}_p) = \frac{(n-3)!}{(n-p-1)!}\left\{\frac{n-2}{2}c^2\omega g - (n-2p)E\right\}. \tag{22}$$

Proof. Let $k \leq p-2$, then using formulas (3),(19) we get

$$\begin{aligned} c^k(\mathcal{N}_p) &= *g^k * (\mathcal{N}_p) = *g^k\mathcal{N}_{n-p} \\ &= *\left\{(\frac{1}{n-p-1}c\omega.g - 2\omega)\frac{g^{n-p-2+k}}{(n-p-2)!}\right\}. \end{aligned}$$

A direct application of formula (15) in [9] shows then that

$$\begin{aligned} c^k(\mathcal{N}_p) = &-\frac{(n-p-1+k)!}{(n-p-1)!}\frac{g^{p-k-1}}{(p-k-1)!}c\omega + \frac{(n-p-1+k)!}{(n-p-1)!}\frac{g^{p-k}}{(p-k)!}c^2\omega \\ &- 2\frac{(n-p-2+k)!}{(n-p-2)!}\frac{g^{p-k-2}}{(p-k-2)!}\omega + 2\frac{(n-p-2+k)!}{(n-p-2)!}\frac{g^{p-k-1}}{(p-k-1)!}c\omega \\ &- \frac{(n-p-2+k)!}{(n-p-2)!}\frac{g^{p-k}}{(p-k)!}c^2\omega. \end{aligned}$$

Note that for $k = p$ (resp. $k = p-1$) the first, third and fourth terms should be dropped from the previous summation (resp. the third term). A direct computation then yields the desired results. ∎

5 Positivity of the Weitzenböck curvatures

Let now $\omega = R$ be the curvature tensor of a Riemannian manifold and $\mathcal{N}_p$ be the corresponding Weitzenböck double form.
The positivity of $\mathcal{N}_p$ is clearly very important in the study of the interactions between the geometry and topology of manifolds. Therefore it is particularly interesting to understand this condition and to relate it to the positivity of the other well known curvatures.
The case $p = 1$ (and by duality $p = n - 1$) is well known since $\mathcal{N}_1$ coincides with the Ricci curvature, so we suppose $2 \leq p \leq n - 2$.
Needless to say at this stage that the positivity of $\mathcal{N}_p$ is strictly stronger than the positivity of its sectional curvature (likewise the positivity of the Riemann curvature operator and of its sectional curvature). We already know that the positivity of the Riemann curvature operator implies the one of $\mathcal{N}_p$ for every p. This is a theorem of Meyer [2], see [10] for a simplified proof of this result.
On the other side, it is clear from equation (20) that the positivity of $\mathcal{N}_p$ implies the positivity of the scalar curvature. Consequently, the positivity of the Weitzenböck curvatures are intermediate between positive scalar curvature and positive curvature operator.

The following proposition is a direct consequence of formulas (20), (21) and (22):

Proposition 5.1 *For each p with $0 \leq p \leq n-1$, a Riemannian n-manifold with positive scalar curvature has its contracted Weitzenböck curvature up to the order p, that is $c^p(\mathcal{N}_p)$, positive.*
Furthermore, in each of the following cases the contracted curvature $c^{p-1}(\mathcal{N}_p)$ is positive:

1. *If $n = 2p + 2$ and (M, g) has positive scalar curvature.*
2. *If $n \leq 2p + 2$ and (M, g) has positive Einstein curvature.*
3. *If $n \geq 2p + 2$ and (M, g) has positive Ricci curvature.*

In what follows, we investigate the relation to the positivity of the p-curvature [4, 5]. Recall that the p-curvature s_p of the Riemann curvature tensor R is defined to be the sectional curvature of the tensor $*\left(\frac{1}{(n-p-2)!}g^{n-p-2}R\right)$.
It is clear that the p-curvature appears naturally in the formula (13) for $\mathcal{N}_p$ but it contributes with a negative sign!. The influence of the positivity of the p-curvature on the positivity of $\mathcal{N}_p$ becomes clear in the following formula:

Proposition 5.2 *For every $2 \leq p \leq n-2$ such that $n+p$ is even we have*

$$\mathcal{N}_{\frac{n+p}{2}} = C(n,p) g^{\frac{n-p}{2}} \left\{ *(\frac{p(p-1)}{(n-p-2)!} g^{n-p-2} R) - \frac{(n-1)(n-2)}{(p-2)!} g^{p-2} W \right\}. \tag{23}$$

Where $C(n,p) = \frac{2(p-2)!}{(\frac{n+p-4}{2})!(n+p-2)(n-p-1)}$ *and* W *is the double form associated to the standard weyl curvature tensor.*

Proof. Let $R = \omega_2 + g\omega_1 + g^2\omega_0$ be the standard decomposition of the Riemann curvature tensor, where $\omega_2 = W$. Applying formula (20) in [9] we obtain

$$*\frac{g^{n-p-2}R}{(n-p-2)!} = \frac{g^{p-2}}{(p-2)!} \left\{ \omega_2 - \frac{n-p-1}{p-1} g\omega_1 + \frac{(n-p)(n-p-1)}{p(p-1)} g^2\omega_0 \right\}.$$

Inserting this in the right hand side of the desired equation one can easily reach to the left hand side. ∎

As a direct consequence of the previous formula we get an alternative proof of our result in [4] on the vanishing of betti numbers around the middle dimension for conformally flat manifolds ($W = 0$) with positive p-curvature. Finally, taking into consideration the existing similarity between positive isotropic curvature and positive p-curvature, see [6, 7], the previous proposition may help in answering the following question:

For a compact n-dimensional Riemannian manifold, does positive isotropic curvature imply the vanishing of the betti numbers b_i for $2 \leq i \leq n-2$?

With reference to the previous proposition, a positive answer to the question above would be possible if one can prove that the positivity of isotropic curvature implies the positivity of the operator

$$*(\frac{p(p-1)}{(n-p-2)!} g^{n-p-2} R) - \frac{(n-1)(n-2)}{(p-2)!} g^{p-2} W,$$

for all $0 \leq p \leq n-4$.

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